[Raj *, 5(11): November 2018]
DOI- 10.5281/zenodo.1569091

ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

CONVERGENCE OF KRASNOSELSKII'S ITERATION OF RELATIVELY NONEXPANSIVE MAPPING IN STRICTLY CONVEX SPACES

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ABSTRACT

Let us consider two nonempty closed convex subsets A, B of a strictly convex space and a mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$ and $T(B) \subseteq A$ and $\|Tx - Ty\| \le \|x - y\|$, for all $x \in A$ and $y \in B$. First, we provided sufficient conditions for the existence of fixed point pairs (x^*, y^*) in $A \times B$ of T for which the distance between x^* and y^* is optimum. It is worth mentioning that, we prove the existence of fixed points without invoking proximal normal structure property[4]. Also, we proved the convergence of krasnoselskii's iteration of a relatively nonexpansive mapping to a fixed point. The main purpose of this article is to provide sufficient conditions to ensure the existence of a pair (x^*, y^*) of points in (A, B) such that $Tx^* = x^*$, $Ty^* = y^*$ for which the distance between the fixed points x^* and y^* is optimum in some sense. It is worth to mentioning that our existence theorem does not relay on the proximal normal structure property. Also, we proved the strong convergence of Krasnoselskii's iteration of T to a fixed point which generalizes a result due to Eldred et.al. [Eldred et.al., Proximal normal structure and relatively nonexpansive mappings, Studia Math. 171(2005),283-293].

AMS Subject Classification: 54H25,47H10

Keywords: *Relatively nonexpansive mappings, P-property, cyclic contraction mappings, fixed points.*

I. INTRODUCTION

Let A be a nonempty subset of a metric space X and $T : A \to A$ be a mapping. Fix $x_0 \in A$. The sequence $\{x_n\}$ of successive iterations of T starting from x_0 is defined as $x_n := T x_{n-1}$, for all $n \in N$. The behaviour of the iterated sequences play an important role in fixed point theory. It is well known fact that if an iterated sequence of a continuous mapping T converges, then the limit of it must be a fixed point of T . Thus, the sequence of successive approximations provides a computational or an algorithmic oriented development to locate the fixed point of the given self mapping.

The well known Banach contraction principle states that every contraction mapping $T : A \to A$, where A is a complete subspace of a metric space X, has unique fixed point in A and every iterated sequence of T starting from any $x \in A$ converges to the unique fixed point of T . But the behaviour of the iterated sequences of nonexpansive mappings are completely different from the iterated sequences of contractive type mappings. It is easy to see examples of nonexpansive mappings having unique fixed point but the iterated sequence need not converge. This situation motivates many researchers to find some alternative approximation to a fixed point of the given nonexpansive mapping. Consider a nonexpansive mapping $T : A \to A$, where A is a nonempty closed convex subset of a normed linear space X. In [1], Krasnoselskii proved that in uniformly convex Banach space X, the sequence of successive approximation of the averaged mapping $F : A \to A$ given by $F(x) := (x + Tx)/2$, for all $x \in A$, converges to a fixed point of the nonexpansive mappings T . A complete proof of Krasnoselskii's results in English can be found in [2]. Later, in [3], Edelstein extended Krasnoselskii's result to strictly convex space setting.

On other hand, in [4], the authors introduced a class of mappings called relatively nonexpansive defined as follows, which generalize the notion of nonexpansive mappings.

Definition 1.1 [4] Let A, B be nonempty subsets of a normed linear space X and $T : A \cup B \to A \cup B$ be a mapping. Then T is said to be a relatively nonexpansive mapping if and only if

1. $T(A) \subseteq A$ and $T(B) \subseteq B$,
2. $\|Tx - Ty\| \leq \|x - y\|$, for all $x \in A$, $y \in B$.

It is worth to note that a relatively nonexpansive mapping need not be continuous, whereas the nonexpansive mappings are uniformly continuous. If $A \cap B$ is nonempty, then the mapping T restricted to $A \cap B$ is a nonexpansive mapping and $T(A \cap B) \subseteq A \cap B$. Let us fix $\text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}$. In [4], the authors provided sufficient conditions for the existence of a pair (x^*, y^*) in $A \times B$ such that $Tx^* = x^*$, $Ty^* = y^*$ and $\|x^* - y^*\| = \text{dist}(A, B)$. The following theorem is one of the main results proved in [4].

Theorem 1.1.[4] Let A, B be nonempty bounded closed convex subsets of a uniformly convex Banach space X . Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping.Then there exists$(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $\|x - y\| = \text{dist}(A, B)$. For fixed $x_0 \in A_0$, and the krasnoselskii's iteration is defined as

$x_n = \frac{(x_{n-1} + Tx_{n-1})}{2}$,for all n∈N.If T(A) lies in a compact subset, then $\{x_n\}$ converges to a fixed point of T .

In [4], the authors introduced and used the geometric notion called proximal normal structure to prove the existence of a point (x, y) in $A \times B$ having the properties mentioned in Theorem 1.1. It has been shown that a pair (A, B) of nonempty bounded closed convex subsets of a uniformly convex Banach space enjoys proximal normal structure. Moreover, the uniform convexity of X is essential to prove the convergence of Krasnoselskii's iteration for relatively nonexpansive mappings.

In this article, we proved a generalised version of Theorem 1.1 in strictly convex Banach space setting. It is worth mentioning that we proved the existence of a point (x, y) in $A \times B$ satisfying the conclusion of Theorem 1.1 without invoking proximal normal structure property. Our results are mainly due to the geometry of strictly convex spaces. Furthermore, we proved that the Krasnoselskii's iteration, defined as in Theorem 1.1, converges strongly to a fixed point of T .

II. PRELIMINARIES

In this section, we introduce few notations and basic definitions, which we used in our main results. For any given pair of subsets A, B of a normed linear space X, define $A_0 := \{x \in A : \|x - y\| = \text{dist}(A, B), \text{ for some } y \in B\}$. In similar way, we can define the subset B_0 of B. It is immediate to see that the set A_0 and B_0 are convex subset of A, B respectively.In [5], the authors provided sufficient conditions which ensure the nonemptiness of the set A_0. Moreover, in [6],the authors proved that the set A_0 is contained in the boundary of the set A .

Definition 2.1 (P -property)[8]Let A, B be nonempty subsets of a normed linear space X. Then the pair (A, B) is said to have d-property if and only if,

$$\left.\begin{array}{l}\|x_1 - y_1\| = \text{dist}(A,B)\\ \|x_2 - y_2\| = \text{dist}(A,B)\end{array}\right\} \implies \|x_1 - x_2\| = \|y_1 - y_2\|,$$

ISSN 2348 – 8034
Impact Factor- 5.070

In [8], the author used P -property to prove the existence of a point $x \in A$ satisfying $\|x - T x\| = dist(A, B)$, where $T : A \to B$. In [10], the authors obtained the following result.

LEMMA 2.1. Let A, B be nonempty closed convex subsets of a strictly convex Banach space X. Then the pair (A, B) has the P-property.

It is worth to note that the converse of Lemma 2.1 is also true. Thus, in [10], the authors obtained a characterization of strictly convex space, by using P -property. Consider a pair (A, B) of nonempty closed convex subsets of a strictly convex Banach space X having $A_0 \neq \Phi$. Then,we can define a function

$P_A : B_0 \to A_0$ having the property that $\|P_A(y) - y\| = dist(A, B)$, forall $y \in B_0$. Now, let us state an important convergence result, which was proved in [11].

LEMMA 2.2. [11]Let A, B be nonempty closed convex sub-sets of a strictly convex Banach space X. Let $\{x_n\}$ be a sequence in A and $y \in B_0$ such that $\|x_n - y\| \to dist(A, B)$. If $\{x_n\}$ is contained in a compact subset of A, then $x_n \to P_A(y)$.

In [9], the authors introduced a new class of mappings called cyclic contraction on metric space setting. For our convenient, we state in normed linear space setting.

Definition 2.2 [9]Let A, B be nonempty subsets of a normed space X. A mapping $T : A \cup B \to A \cup B$ is to said be a cyclic contraction mapping if and only if there exists $k \in [0, 1)$ such that

1. $T(A) \subseteq A$ and $T(B) \subseteq B$,
2. $\|T x - T y\| \leq k\|x - y\| + (1 - k)dist(A, B)$, for all $x \in A$, $y \in B$.

It is easy to see that cyclic contraction mappings are relatively nonexpansive mappings. Our main idea is to approximate the given relatively nonexpansive mappings, by means of cyclic contraction mappings.

III. MAIN RESULTS

Let us begin with the proposition.

Proposition 3.1. Let A, B be nonempty closed convex subsets of a strictly convex Banach space X. Then A_0 is a closed subset of A.
PROOF. If A_0 is empty, then nothing to prove. Assume that A_0 is nonempty. Let $\{x_n\}$ be a seqeuence in A_0 such that $x_n \to x_0$, for some x_0 in A. Since $x_n \in A_0$, there exists $y_n \in B$ such that, $\|x_n - y_n\| = dist(A, B)$, for all $n \in N$. By the P- property of strictly convex space, $\|x_n - x_m\| = \|y_n - y_m\|$, for all $n, m \in N$. Since $\{x_n\}$ converges, $\{y_n\}$ is a Cauchy sequence in B, and hence converges. Say $y_n \to y_0$, for some $y_0 \in B$. Thus $\|x_n - y_n\| \to \|x_0 - y_0\|$. Since $\{\|x_n - y_n\|\}$ is a constant sequence, $\|x_0 - y_0\| = dist(A, B)$ and hence $x_0 \in A_0$. Thus A_0 is a closed subset of A.
It is well known fact that a relatively nonexpansive mapping need not be continuous. But the following lemma shows that a relatively nonexpansive mapping restricted to a suitable domain behaves like a continuous mapping.

LEMMA 3.1. Let A, B be nonempty closed convex subsets of a strictly convex space X and $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose T (A) is contained in a compact subset of A. Then T is continuous on A_0.
PROOF. Let $x_0 \in A_0$ and $\{x_n\}$ be a sequence in A such that $x_n \to x_0$. Then there exists unique $y_0 \in B_0$ such that $\|x_0 - y_0\| = dist(A, B)$. Since T is relatively nonexpansive, $\|T x_0 - T y_0\| = dist(A, B)$. This shows that $P_A(T y_0) = T x_0$ Now,let us show that $T(x_n) \to T(x_0)$. By invoking Lemma 2.2, it is enough to show that $\|T x_n - T y_0\| \to dist(A, B)$. Note that $dist(A, B) \leq \|T x_n - T y_0\| \leq \|x_n - y_0\| \leq \|x_n - x_0\| + \|x_0 - y_0\| \to dist(A,B)$

[Raj *, 5(11): November 2018] ISSN 2348 – 8034
DOI- 10.5281/zenodo.1569091 Impact Factor- 5.070

Hence, by Lemma 2.2, $T x_n \to P_A(T y_0) = T x_0$. Thus T is continuous at x_0. Since $x_0 \in A_0$ is arbitrary, T is continuous on A_0.

Let (A, B) be a pair of subsets of a normed linear space X and T: $A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Then it is easy to verify that $T(A_0) \subseteq A_0$ and $T(B_0) \subseteq B_0$. The following theorem provides sufficient conditions to ensure the existence of fixed points $x \in A$ and $y \in B$ of a cyclic contraction mapping T on $A \cup B$, satisfying $\|x - y\| = \mathrm{dist}(A, B)$.

Theorem 3.1. Let A, B be nonempty, closed, convex subsets of a strictly convex Banach space X such that A_0 is nonempty. Let $T:A \cup B \to A \cup B$ be a cyclic contraction mapping. Suppose T (A) is contained in a compact subset of A. Then there exists $(x_n, y_n) \in A \times B$ such that $T x_n = x_n$, $T y_n = y_n$ and $\| x_n - y_n \| = \mathrm{dist}(A, B)$.
PROOF. Choose $x_0 \in A_0$.Then there exists $y_0 \in B_0$ such that $\|x_0 - y_0\| = \mathrm{dist}(A, B)$. Let $x_n := T x_{n-1}$ and $y_n := T y_{n-1}$, for all $n \in N$. The sequences $\{x_n\}$, $\{y_n\}$ are in A_0, B_0 respectively, such that $\|x_n - y_n\| = \mathrm{dist}(A, B)$, for all $n \in N$. Also, it is easy to verify that $\mathrm{dist}(A, B) \le \|T x_n - y_n\| \le k^n\|x_1 - y_0\| + (1 - k^n)\mathrm{dist}(A, B) \to \mathrm{dist}(A, B)$

Thus, $\|Tx_n - y_n \| \to \mathrm{dist}(A,B)$. Since $T(A_0)$ is contained in a compact subset of A, $\{ x_n \}$ has a convergent subsequence $\{ x_{nk} \}$, say $x_{nk} \to x^*$. Since A_0 is closed, $x^* \in A_0$. Then there is $y^* \in B$ such that

$$\|Tx_* - Ty_*\| = \|x_* - y_*\| = \mathrm{dist}(A,B).$$

Since $\| y_{nk} - x_* \| \le \| y_{nk} - x_{nk} \| + \| x_{nk} - x_* \| \to \mathrm{dist}(A, B)$ and by Lemma2.2, $y_{nk} \to P_B(x^*) = y^*$. Since T is continuous at x^*, $T x_{nk} \to Tx^*$. Then, $\mathrm{dist}(A, B) \le \|Tx^* - y^*\| \le \|T x^* - T x_{nk}\| + \|T x_{nk} - y_{nk}\| +$
$\| y_{nk} - y^*\| \to \mathrm{dist}(A, B)$.

This shows that $\|Tx^* - y^*\| = \mathrm{dist}(A, B)$. By P-property, we conclude that $Tx^* = x^*$ and in similar manner, we can show that $Ty^* = y^*$.

Theorem 3.2. Let A, B be nonempty, closed, convex subsets of a strictly convex Banach space X such that A_0 is nonempty. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose T (A) is contained in a compact subset A_1 of A. Then there exists $(x^*, y^*) \in A \times B$ such that $T x^* = x^*$, $T y^* = y^*$ and
$\|x^* - y^*\| = \mathrm{dist}(A, B)$.
PROOF. Choose $(x_0, y_0) \in A_0 \times B_0$ such that $\|x_0 - y_0\| = \mathrm{dist}(A, B)$. For each $n \in N$, consider the mapping
$T_n : A \cup B \to A \cup B$ defined by

$$T_n(x) := \begin{cases} \frac{1}{n} x_0 + (1 - \frac{1}{n}) T_x, & \text{if } x \in A \\ \frac{1}{n} y_0 + (1 - \frac{1}{n}) T_x, & \text{if } x \in B \end{cases}$$

For fixed $n \in N$, the convexity of A and B shows that $T_n(A) \subseteq A$ and $T_n(B) \subseteq B$. Also, it is easy to verify that

$$\| x_n - T(x_n) \| = \frac{1}{n} \| x_0 - T(x_n) \| \le \frac{1}{n} M$$

where $M > 0$ is an upper bound for $\|x_0 - T(x_n)\|$. This concludes that $\|x_n - T(x_n)\| \to 0$, as $n \to \infty$. Since $\{x_n\}$ is a seqeuence in the compact subset $co(\{x_0\} \cup A_1)$ of A the sequence $x_n = T_n(x_n)$ has a convergent subsequence$\{ x_{nk} \}$, say $x_{nk} \to x^*$, for some $x^* \in A$. Since $x_{nk} \in A_0$ and A_0 is closed subset of A (by Theorem3.1), $x^* \in A_0$. Then there exist $y^* \in B_0$ such that $\|x^* - y^*\| = \mathrm{dist}(A, B)$. By using P-property, it is easy to see that $y_{nk} \to y^*$. i.e., $\|y_{nk} - y^*\| = \|x_{nk} - x^*\| \to 0$. Now, let us show that $x_{nk} \to T(x^*)$, which concludes that $T(x^*) = x^*$. By Lemma2.2, it is enough to show that $\|x_{nk} - T(y^*)\| \to \mathrm{dist}(A, B)$. Consider the following inequality.

ISSN 2348 – 8034
Impact Factor- 5.070

$$\begin{aligned}\operatorname{dist}(A,B) &\le \|x_{n_k} - T(y^*)\| = \|T_{n_k}(x_{n_k}) - T(y^*)\| \\ &\le \tfrac{1}{n_k}\|x_0 - Ty^*\| + (1-\tfrac{1}{n_k})\|Tx_{n_k} - Ty^*\| \\ &\le \tfrac{1}{n_k}\|x_0 - Ty^*\| + (1-\tfrac{1}{n_k})\|x_{n_k} - y^*\| \\ &\le \tfrac{1}{n_k}\|x_0 - Ty^*\| + (1-\tfrac{1}{n_k})[\|x_{n_k} - x^*\| + \|x^* - y^*\|] \\ &\to \operatorname{dist}(A,B).\end{aligned}$$

Hence, by Lemma2.2, $x_{nk} \to P_A(T(y^*)) = T(x^*)$. Thus, $T(x^*) = x^*$. In similar manner, we can show that $T(y^*) = y^*$.

The following theorem provides sufficient conditions for the strong convergence of the Krasnoselskii's iterations of a relatively nonexpansive mapping T defined on $A \cup B$.

Theorem 3.3. Let A, B be nonempty closed convex subsets of a strictly convex Banach space X such that A_0 is nonempty. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose T (A) is contained in a compact subset A_1 of A. Then the Krasnoselskii's iteration $\{F^n(x)\}$, where $F : A \cup B \to A \cup B$ given by $F(x) = 1/2\,(Tx + x)$, converges to a fixed point of T .

PROOF. Let $x \in A_0$ and $y \in B_0$ such that $T(x) \neq x$ and $T(y) = y$.

Note that by Theorem3.2, the fixed point of T in B exists. Since $\|T(x) - y\| \le \|x - y\|$ and $\|x - y\| \le \|x - y\|$, the strict convexity of the norm implies $\left\|\frac{T(x)+x}{2} - y\right\| < \|x - y\|$. That is, for all $x \in A$, $y \in B$, satisfying $T(x) \neq x$, $T(y) = y$, we have

$$\|F(x) - y\| < \|x - y\| \qquad (2)$$

It is worth to note that the set $\{F^n(x) : n \in N\}$ is a subset of the closed convex hull of $A_1 \cup \{x\}$. Since $\overline{co(\{x\} \cup A_1)}$ is compact, $\{F^n(x)\}$ has a subsequence $\{F^{ni}(x)\}$ such that $F^{ni}(x) \to p$, for some $p \in A$. Since A_0 is convex, the sequence $\{F^n(x)\}$ is in A_0 and consequently, the closedness of A_0 assures that $p \in A_0$. Then there is unique $q \in B_0$ such that $\|p - q\| = \operatorname{dist}(A, B)$.

For a moment, let us assume that $T(p) = p$. Then we show that $F^n(x) \to p$ as $n \to \infty$. By invoking Lemma2.2, it is enough to show that $\|F^n(x) - q\| \to \operatorname{dist}(A, B)$. Since T is relatively nonexpansive,it is easy to see that $\|p - T(q)\| = \operatorname{dist}(A, B)$. Hence, by strict convexity, $T(q) = q$ and it follows $F(q) = q$. The relatively nonexpansive property of F shows that $\|F^{n+1}(x) - q\| \le \|F^n(x) - q\|$, for all $n \in N$. Thus, $\{\|F^n(x) - q\|\}$ is a monotonically decreases sequence of non-negative real number and hence it converges, say $\|F^n(x) - q\| \to r$, for some $r \ge 0$. Since the subsequence $\|F^{ni}(x) - q\| \to \|p - q\| = \operatorname{dist}(A, B)$, we conclude that $\|F^n(x) - q\| \to \operatorname{dist}(A, B)$ and hence, by Lemma2.2, $F^n(x) \to P_A(q) = p$.

Thus the proof will complete, if we show that $T(p) = p$.

Suppose $T(p) \neq p$. Then, we show that no element in $\{F^n(x)\}$ can be fixed under T . If not, there exists $k \in N$ such that $T(F^k(x)) = F^k(x)$. Then by definition of F , it is easy to verify that $F^{k+i}(x) = F^k(x)$, for all $i \in N$. Thus $F^n(x)$ is an evantually constant sequence converges to $F^k(x)$, and hence $F^k(x) = p$. This shows that $T(p) = p$, a contradiction.

Since $T(y) = y$ and $T(F^n(x)) \neq F^n(x)$, for all $n \in N$, by recursive use of (2), we have

$$\|F^{n+i}(x) - y\| < \|F^n(x) - y\|, \text{ for all } i \in N \qquad (3)$$

Consider the open ball $B(F(p), r)$, where $r = 1/2(\|p-y\| - \|F(p) - y\|)$. Since $T(p) \neq p$, by (2), $r > 0$.Since F is a relatively nonexpansive mapping and $p \in A_0$, F is continuous at p. Hence there exists an open ball $B_1(p, \rho)$ in A_0 such that $F(B_1(P, \rho)) \subseteq B(F(p), r)$.

Since $F^{ni}(x) \to p$, there exists $k \in N$ such that $F^k(x) \in B(P, \rho)$, and hence $F^{k+i}(x) \in B(F(p), r)$. Then

[Raj *, 5(11): November 2018] ISSN 2348 – 8034
DOI- 10.5281/zenodo.1569091 Impact Factor- 5.070

$$\begin{aligned}\|F^{k+i}(x) - y\| &< \|F^{k+1}(x) - y\| \quad (by3)\\ &\leq \|F^{k+1}(x) - F(p)\| + \|F(p) - y\|\\ &< r+\|F(p)-y\| = \tfrac{1}{2}(\|p-y\|+\|F(p)-y\|)\end{aligned}$$

Then, for any $i \in \mathbb{N}$,

$$\begin{aligned}\|F^{k+i}(x) - p\| &\geq \|y - p\| - \|F^{k+i}(x) - y\|\\ &> \|y - p\| - \tfrac{1}{2}(\|p - y\| + \|F(p) - y\|) = r.\end{aligned}$$

This shows that no subsequence of $\{F^n(x)\}$ converges to p, a contradiction. Hence T (p) = p and this completes the proof.

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